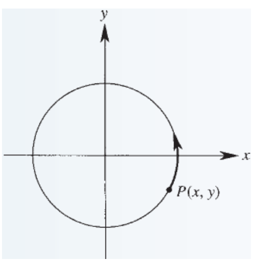
Topics Covered:

Locus (Functions)	Chapter 5 and 10
Plane Geometry	Chapter 4
Linear Functions	Chapter 7
Differentiation (Intro to Calculus)	Chapter 8

Locus & the Parabola

A locus is the term used to describe the path of a single moving point that obeys certain conditions.



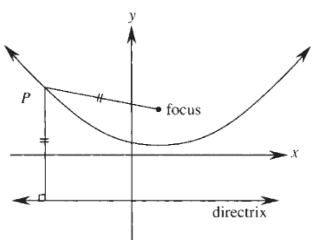
Circle as a Locus

The locus of point *P(x, y)* that is always a constant distance from a fixed point is a circle. The circle, centre *(0, 0)* and radius *r* has the equation: $x^2 + y^2 = r^2$

The circle, centre *(a, b)* and radius *r* has the equation: $(x-a)^2 + (y-b)^2 = r^2$

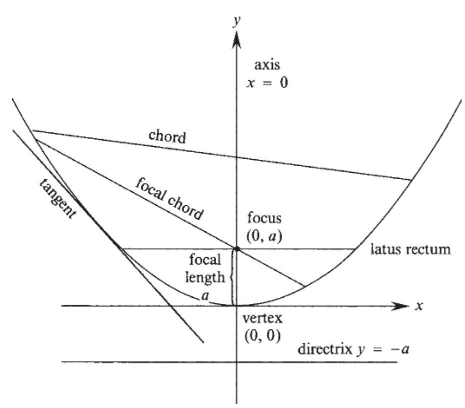
Parabola as a Locus

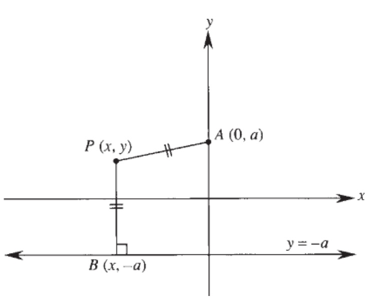
The locus of a point that is equidistant from a fixed point and a fixed line is always a parabola. The fixed point is called the focus and the fixed line is called the directrix.



A parabola is equidistant from a fixed point and a fixed line.

- The fixed point is called the focus.
- The fixed line is called the directrix.
- The turning point of the parabola is called the vertex.
- The axis of symmetry of the parabola is called its axis.
- The distance between the vertex and the focus is called the focal length.
- An interval joining any two points on the parabola is called a chord.
- A chord that passes through the focus is called a focal chord.
- The focal chord that is perpendicular to the axis is called the latus rectum.
- A tangent is a straight line that touches the parabola at a single point.





PARABOLA $x^2 = 4ay$

The locus of point *P(x, y)* moving so that it is equidistant from the point *(0, a)* and the line *y = -a* is a parabola with equation. $\boldsymbol{x^2 = 4ay}$

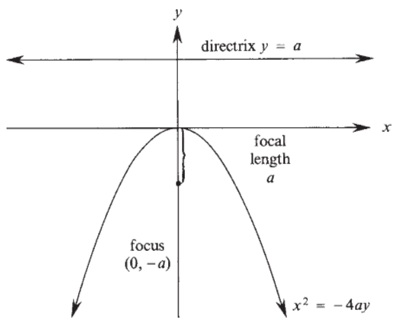
The parabola $x^2 = 4ay$ has

- focus at *(0,a)*

- directrix with equation *y = -a*
- Vertex at *(0,0)*
- Axis with equation *x = 0*
- Focal length the distance from the vertex to the focus with length *a*
- latus rectum that is a horizontal focal chord with length *4a*

PARABOLA $x^2 = -4ay$

The locus of point *P(x, y)* moving so that it is equidistant from the point *(0, -a)* and the line *y = -a* is a parabola with equation. $x^2 = -4ay$

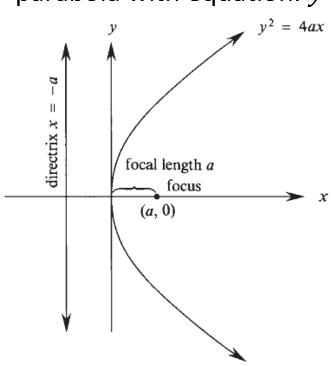


The parabola $x^2 = -4ay$ has

- focus at *(0, -a)*
- Directrix with equation *y = a*
- Vertex at *(0, 0)*
- Axis with equation *x = 0*
- Focal length *a*
- latus rectum a horizontal focal chord with length *4a*

PARABOLA $y^2 = 4ax$

The locus of point p(x, y) moving so that it is equidistant from the point (a, 0) and the line x = -a is a parabola with equation. $y^2 = 4ax$



The parabola $y^2 = 4ax$ has

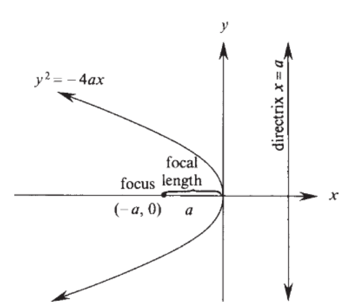
- Focus at *(a, 0)*
- Equation of directrix *x = -a*
- Vertex at *(0,0)*
- Axis with equation *y = 0*
- Focal length the distance from the vertex to the focus with length *a*
- Latus rectum that is a vertical focal chord with length *4a*

PARABOLA $y^2 = -4ax$

The locus of a point P(x, y) moving so that it is equidistant from the point *(-a, 0)* and the line *x = a* is a parabola with equation. $y^2 = -4ax$

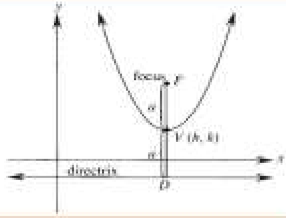
The parabola $y^2 = -4ax$ has

- Focus at *(-a , 0)*
- Directrix with equation *x = a*
- Vertex at *(0, 0)*
- Axis with equation *y = 0*
- Focal length *a*
- Latus rectum a vertical focal chord with length *4a*



PARABOLA $(x-h)^2 = 4a(y-k)$

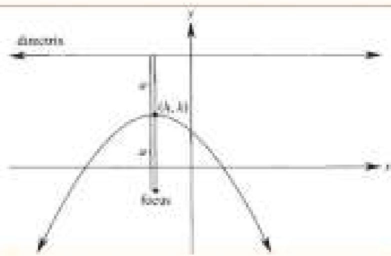
The concave upwards parabola with vertex (h, k) and focal length a has equation $(x-h)^2 = 4a(y-k)$



The parabola $(x-h)^2 = 4a(y-k)$ has
- **axis** parallel to the y-axis
- **vertex** at (h, k)
- **focus** at $(h, k+a)$
- **directrix** with equation $y = k - a$

PARABOLA $(x-h)^2 = -4a(y-k)$

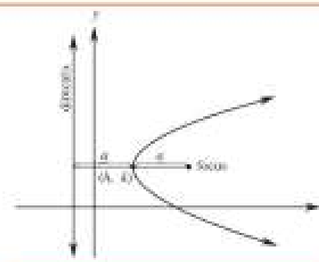
The concave downwards parabola with vertex (h, k) and focal length a has equation $(x-h)^2 = -4a(y-k)$



The parabola $(x-h)^2 = -4a(y-k)$ has
- **axis** parallel to the y-axis
- **vertex** at (h, k)
- **focus** at $(h, k-a)$
- **directrix** with equation $y = k + a$

PARABOLA $(y-k)^2 = 4a(x-h)$

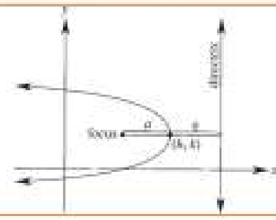
The parabola with vertex (h, k) and focal length a that turns to the right has equation $(y-k)^2 = 4a(x-h)$



The parabola $(y-k)^2 = 4a(x-h)$ has
- **axis** parallel to the x-axis
- **vertex** at (h, k)
- **focus** at $(h+a, k)$
- **directrix** with equation $x = h - a$

PARABOLA $(y-k)^2 = -4a(x-h)$

The parabola with vertex (h, k) and focal length a that turns to the left has equation $(y-k)^2 = -4a(x-h)$



The parabola $(y-k)^2 = -4a(x-h)$ has
- **axis** parallel to the x-axis
- **vertex** at (h, k)
- **focus** at $(h-a, k)$
- **directrix** with equation $x = h + a$

Tangents and Normals

Remember that the gradient of the tangent to a curve is given by the derivative. The normal to the curve is perpendicular to its tangent at that point. That is, $m_1m_2 = -1$ for perpendicular lines.

Plane Geometry

COMMON ABBREVIATIONS

∠ ⇒ angle	isos. ⇒ isosceles
Δ ⇒ triangle	equil. ⇒ equilateral
quad. ⇒ quadrilateral	vert.opp. ⇒ vertically opposite
// ⇒ is parallel to	corresp. ⇒ corresponding
⊥ ⇒ is perpendicular to	alt. ⇒ alternate
// gram ⇒ parallelogram	co - int. ⇒ co - interior
str. line ⇒ straight line	ext. ∠ ⇒ exterior angle
opp. ⇒ opposite	int. ∠'s ⇒ interior angles
pt. ⇒ point	

Type of Angle	Description
Acute	Any angle less than 90°
Right Angle	An angle that is 90° exactly
Obtuse Angle	An angle that is greater than 90° but less than 180°
Straight Angle	An angle that is 180° exactly
Reflex Angle	An angle that is greater than 180°
Full Rotation	360°

The point is called B.

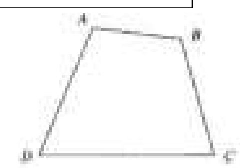
The interval (part of a line) is called AB or BA.

If AB and CD are parallel lines, we write $AB \parallel CD$.

This angle is named $\angle BAC$ or $\angle CAB$. It can sometimes be named $\angle A$.

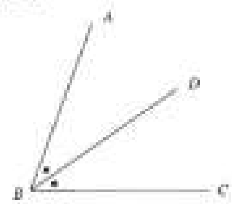
Angles can also be written as $B\hat{A}C$ or $\underline{/BAC}$.

This triangle is named ΔABC.

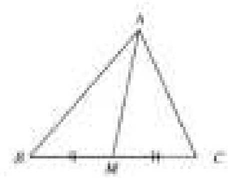


This quadrilateral is called $ABCD$.

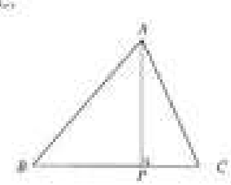
Line AB is produced to C.



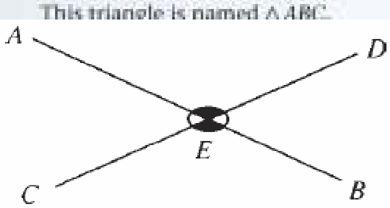
DB bisects $\angle ABC$.



AM is a median of ΔABC.

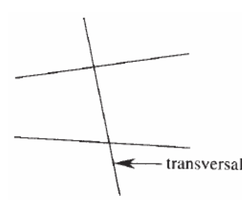


AP is an altitude of ΔABC.

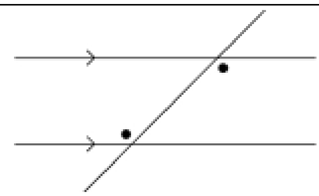
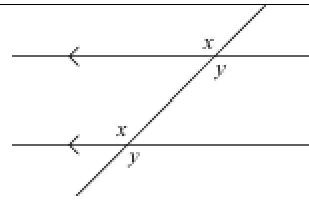
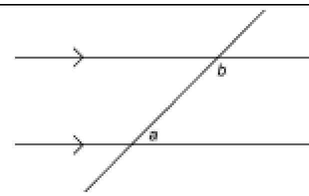


Vertically Opposite Angles are Equal

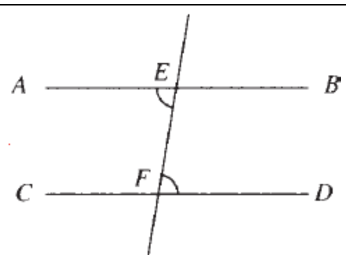
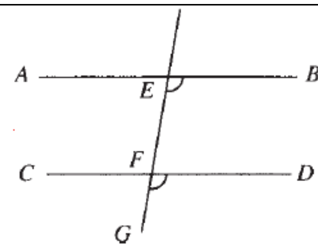
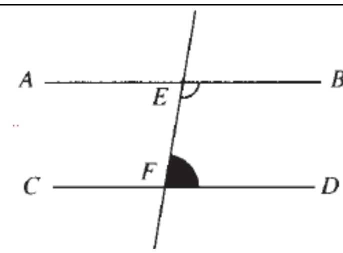
Parallel Lines



When a transversal cuts two lines, it forms pairs of angles. When the two lines are parallel, these pairs of angles have special properties.

Alternate Angle If the lines are parallel, then alternate angles are equal.	**Corresponding Angle** If the lines are parallel, then corresponding angles are equal.	**Cointerior Angle** If the lines are parallel, cointerior angles are supplementary (i.e. their sum is 180°)
		

Tests for Parallel Lines

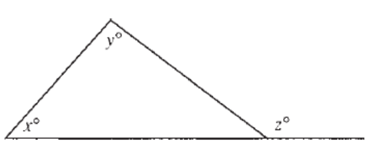
If alternate angles are equal, then the lines are parallel.	**If corresponding angles are equal, then the lines are parallel.**	**If cointerior angles are supplementary, then the lines are parallel.**
		

If 2 lines are both parallel to a third line, then the 3 lines are parallel to each other. That is, if AB || CD and EF || CD, then AB || EF.

Types of Triangles

- A scalene triangle has no two sides or angles equal.
- A right (or right-angled) triangle contains a right angle.
- An isosceles triangle has two equal sides.
- The angles (called the base angles) opposite the equal sides in an isosceles triangle are equal.
- An equilateral triangle has three equal sides and angles.
- An obtuse-angled triangle contains an obtuse angle.
- Interior Angle Sum of any triangle is 180°

Exterior Angle of a Triangle



x + y = z

Congruency Tests - Two triangles are congruent if

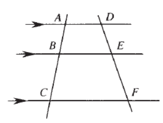
- SSS: all three pairs of corresponding sides are equal
- SAS: two pairs of corresponding sides and their included angles (angle between the 2 sides) are equal

- AAS: two pairs of angles and one pair of corresponding sides are equal
- RHS: both have a right angle, their hypotenuses are equal and one other pair of corresponding sides are equal

Similarity Tests – Two triangle are similar if

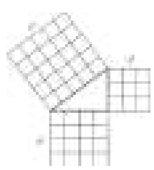
- Three pairs of corresponding angles are equal (If 2 pairs of angles are equal then the third pair must also be equal)
- Three pairs of corresponding sides are in proportion
- Two pairs of sides are in proportion and their included angles are equal

Ratio of Intercepts



When two (or more) transversals cut a series of parallel lines, the ratios of their intercepts are equal. That is, AB: BC = DE: EF or $\frac{AB}{BC} = \frac{DE}{EF}$

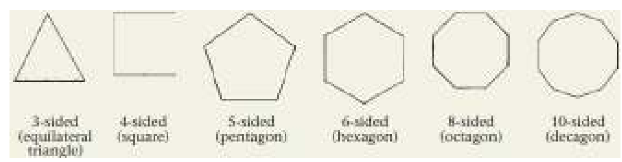
Pythagoras Theorem



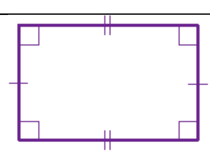
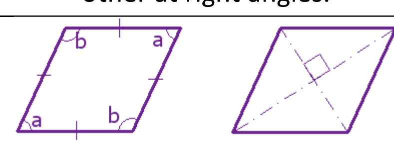
The square on the hypotenuse in any right-angled triangle is equal to the sum of the squares on the other two sides. That is, $a^2 + b^2 = c^2$ or $c = \sqrt{a^2+b^2}$

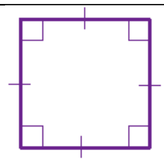
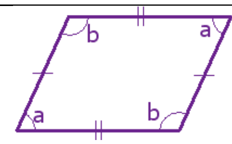
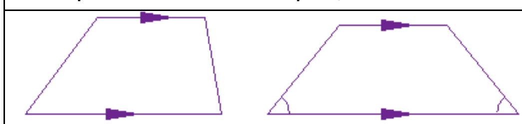
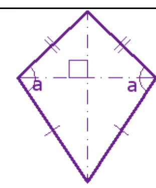
If $a^2 + b^2 = c^2$, then ABC must be right angled.

Types of Quadrilaterals



- A quadrilateral is any four-sided figure.
- In any quadrilateral the sum of the interior angles is 360^o
- A polygon is a closed plane figure with straight sides.
- A regular polygon has all sides and all interior angles equal.

RECTANGLE [*A= lb*] A rectangle is a four-sided shape where every angle is a right angle (90°). Also opposite sides are parallel and of equal length.	RHOMBUS [*A= ½xy*] A rhombus is a four-sided shape where all sides have equal length. Also opposite sides are parallel and opposite angles are equal. Another interesting thing is that the diagonals (dashed lines in second figure) of a rhombus bisect each other at right angles.
	 x & y are the diagonals
SQUARE [*A=lb*] A square has equal sides and every angle is a right angle (90°) Also opposite sides are parallel. A square also fits the definition of a rectangle (all angles are 90°), and a rhombus (all sides are equal	PARALLELOGRAM [*A = bh*] A square has equal sides and every angle is a right angle (90°) Also opposite sides are parallel. A square also fits the definition of a rectangle (all angles are 90°), and a rhombus (all sides are equal length).

length).	
	
TRAPEZIUM [$A = ½h(a+b)$] A trapezium has one pair of opposite sides parallel. It is called an Isosceles trapezium if the sides that aren't parallel are equal in length and both angles coming from a parallel side are equal, as shown.	KITE It has two pairs of sides. Each pair is made up of adjacent sides that are equal in length. The angles are equal where the pairs meet. Diagonals (dashed lines) meet at a right angle, and one of the diagonal bisects (cuts equally in half) the other.
	

Polygons

A polygon is a closed plane figure with straight sides.

A regular polygon has all sides and all interior angles equal.

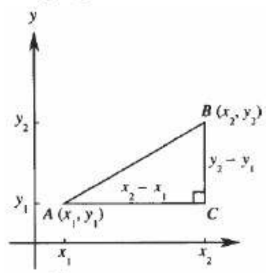
The sum of the interior angles of an n -sided polygon is given by: $S = 180n – 360$ or $S = (n – 2) \times 180^o$ The sum of the exterior angles of any polygon is 360^o

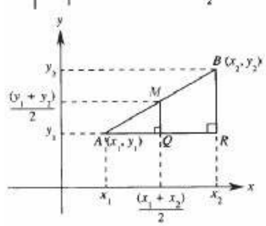
Area of a Circle $A = \pi r^2$

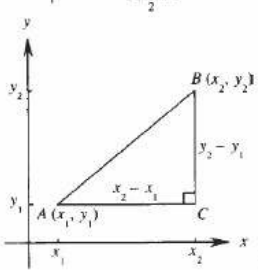
Roydon Ng

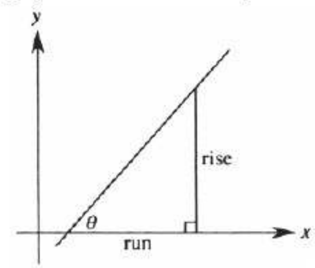
CHAPTER 7: LINEAR FUNCTIONS

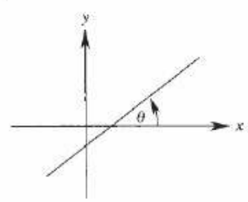
1. Distance between (x_1, y_1) and (x_2, y_2) is given by: $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$
2. Midpoint of two points (x_1, y_1) and (x_2, y_2) is given by: $M = (\frac{x1+x2}{2}, \frac{y1+y2}{2})$
3. Positive gradient leans to the right. / Negative gradient leans to the left. \
4. Gradient $= \frac{rise}{run} = m = \frac{y2-y1}{x2-x1} = \tan\theta$ (where theta is the angle the line makes with the x-axis in a positive direction)
5. Gradient Form: $y = mx + b$ where m = gradient and b = y-intercept
6. The gradient form of the line $ax + by + c = 0$ is given by: $m = -\frac{a}{b}$
7. Intercept Form: $\frac{x}{a} + \frac{y}{b} = 1$ (where *a* and *b* are the *x-int* and *y-int* respectively)
8. Point Gradient formula: the equation of a straight line is given by $y-y1 = m(x-x_1)$ where (x_1, y_1) lies on the line with gradient *m*.
9. Two point formula: the equation of a straight line is given by $\frac{y-y1}{x-x1} = \frac{y2-y1}{x2-x1}$ (where (x_1, y_1) and (x_2, y_2) are points on the line)
10. If two lines are parallel, then they have the same gradient. That is, $m_1 = m_2$ Two lines that are parallel have equations $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$
11. If two lines with gradients m_1 and m_2 respectively are perpendicular, then $m1m2=-1$ (i.e. $m2 = -\frac{1}{m1}$)
12. Perpendicular lines have equations in the form $ax + by + c_1 = 0$ and $bx - ay + c_2 = 0$
13. If $a_1x + b_1y + c_1 = 0$ and $a_2x + b2_y + c_2 = 0$ are 2 given lines then equation of a line through their intersection is given by the formula $(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2)$ where k is a constant.
14. The perpendicular distance from $(x1, y1)$ to the line $ax + by + c = 0$ is given by $d = \frac{|ax+by1+c|}{\sqrt{a^2+b^2}}$

1. 

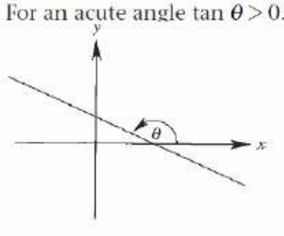
2. 

3.  See above notes

4. 

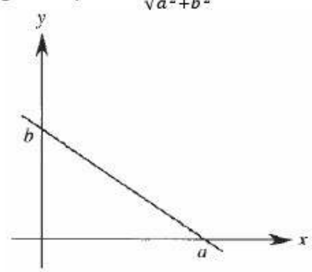


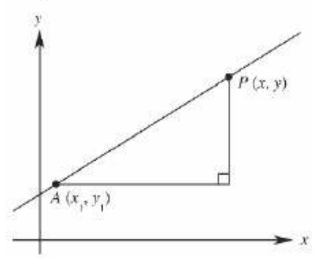
For an acute angle $\tan\theta > 0$.

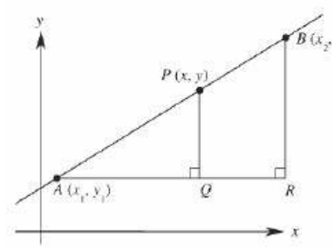


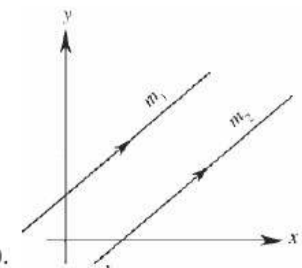
For an obtuse angle $\tan\theta < 0$

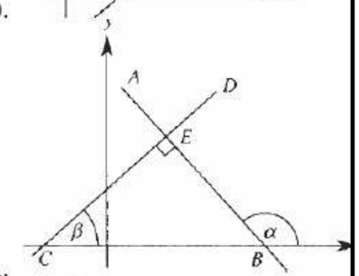
5. See above notes
6. See above notes

7. 

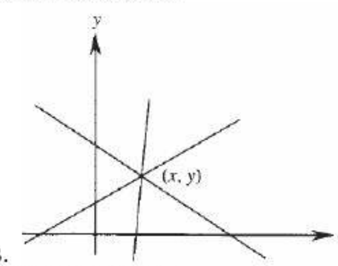
8. 

9. 

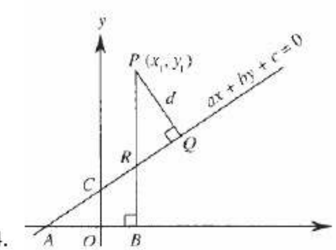
10. 

11. 

12. See above notes

13. 

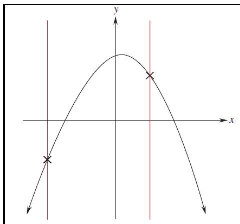
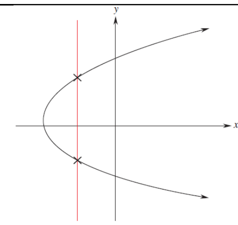
See above notes

14. 

Functions and Graphs

A **function** is a special type of relation. It is like a machine where for every INPUT there is only one OUTPUT.

A **relation** is a set of ordered points (*x , y*) where the variables x and y are related according to some rule.

	If a vertical line cuts a graph only once anywhere along the graph, the graph is a function.		If a vertical line cuts a graph in more than one place anywhere along the graph, the graph is not a function.

Function notation

If *y* depends on what value we give *x* in a function, then we can say that *y* is a function of *x* . We can write this as y = *f(x)*. If *y = f (x)* then *f (a)* is the value of *y* at the point on the function where *x = a*

Intercepts

One of the most useful techniques is to find the x- and y -intercepts. For x -intercept, *y = 0* For y -intercept, *x = 0*

Domain and Range

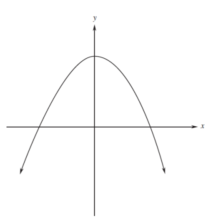
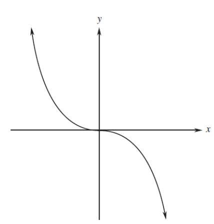
The *x -coordinate* is called the independent variable and the *y -coordinate* is the dependent variable.

The set of all real numbers *x* for which a function is defined is called the domain. The set of real values for *y* or *f(x)* as *x* varies is called the range (or image) of *f* .

Odd and Even functions

If a curve is increasing, as x increases, so does y , and the curve is moving upwards, looking from left to right.

If a curve is decreasing, then as x increases, y decreases and the curve moves downwards from left to right.

	For even functions, *f*(*x*) = *f* (*-x*) for all values of *x* .		Functions are odd if they have point symmetry about the origin. A graph rotated 180° about the origin gives the original graph. For odd functions, f (-x) = -f (x) for all values of x in the domain.

For the family of functions *y = k f* (*x*) , as *k* varies, the function changes its slope or steepness.

For the family of functions *y = f (x)+ k*, as *k* varies, the graph moves up or down (vertical translation).

For the family of functions *y = f* (*x + k)* , as *k* varies, the graph moves left or right (horizontal translation).

Linear Functions (straight lines)

Gradient form: *y* = *mx* + *b* has gradient *m* and *y* -intercept *b*

General form: *ax* + *by* + *c* = 0

The linear function *ax* + *by* + *c* = 0 has domain {all real *x* } and range {all real *y* } where *a* and *b* are non-zero.

x = a is a vertical line with x -intercept a Domain: {x: x = a} Range: {all real y }
y = b is a horizontal line with y -intercept b Domain: {all real x } Range: {y: y = b}

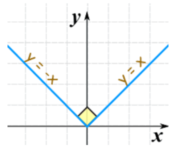
Quadratic Functions

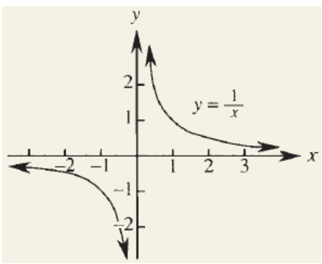
f(x) = ax^2 *+ bx +c* is the general equation of a parabola. If a a>0 the parabola is concave upwards (U shaped). If a a<0 the parabola is concave downwards (^ shaped).

Absolute Value Functions

Use table of values e.g. -3 ≤ x ≤ 3 or definition of absolute value.

$$f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$





Hyperbola

Equation: *xy = a* or $y=\frac{a}{x}$

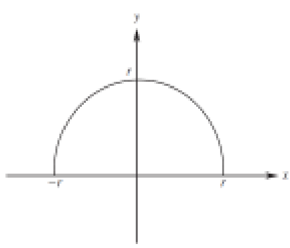
x cannot be = 0 (undefined)

The function $f(x) = \frac{a}{bx + c}$ is a hyperbola with domain $\left\{\text{all real } x\text{: } x \neq -\frac{c}{b}\right\}$ and range {all real y: $y \neq 0$}

Circles

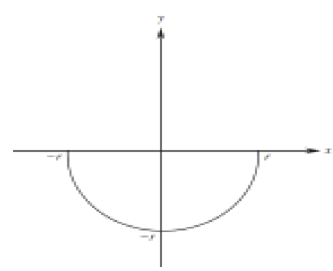
A graph whose equation is in the form $x^2 + ax + y^2\ by + c = 0$ has the shape of a circle.
There is a special case of this formula: The graph of $x^2 + y^2 = r^2$ is a circle, centre (0, 0) and radius *r*
The circle x2 + y2 = r2 has domain: {x: -r ≤ x ≤ r} and range: {x: -r ≤ y ≤ r}
The equation of a circle, centre (a , b) and radius r is $(x – a)^2\ (y – b)^2 = r^2$

$y = \sqrt{r^2 - x^2}$ is the semi-circle above the x-axis since its range is $y \geq 0$ for all values.



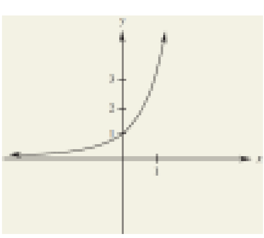
The domain is {x: $-r \leq x \leq r$} and the range is {y: $0 \leq y \leq r$}

$y = -\sqrt{r^2 - x^2}$ is the semi-circle ~~above~~ below the x-axis since its range is $y \leq 0$ for all values.



The domain is {x: $-r \leq x \leq r$} and the range is {y: $-r \leq y \leq 0$}

Exponentials

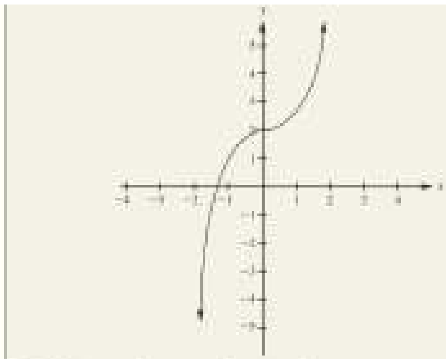


From the graph, *x* can be any real value (the equation shows this as well since any *x* value substituted into the equation will give a value for *y*). From the graph, *y* is always positive, which can be confi rmed by substituting different values of *x* into the equation. Domain: all real *x* Range: y>0

The exponential function $y = a^x$ has domain {all real x} and range {y: $y > 0$}

Cubic Function

A cubic function has an equation where the highest power of x is x^3.



The function can have any real x or y value:
Domain: {all real x}
Range: {all real y}

When finding the domain, we look for values of x that are impossible. For example, with the hyperbola you have already seen that the denominator of a fraction cannot be zero. For the range, we look for the results when different values of x are substituted into the equation. For example, x^2 will always be zero or a positive number.

Limits

The exponential function and the hyperbola are examples of functions that approach a limit. The curve $y = a^x$ approaches the x-axis when x approaches very large negative numbers, but never touches it.

That is, when $x \to -\infty, a^x \to 0$.

Putting $a^{-\infty}$ into index form gives

$$a^{-\infty} = \frac{1}{a^{\infty}}$$
$$= \frac{1}{\infty}$$
$$\doteqdot 0$$

A line that a graph approaches but never touches is called an ***asymptote****.*

We say that the limit of a^x as x approaches $-\infty$ is 0. In symbols, we write $\lim_{x \to -\infty} a^x = 0$.

Continuity

Many functions are continuous. That is, they have a smooth, unbroken curve (or line). However, there are some discontinuous functions that have gaps in their graphs. The hyperbola is an example. If a curve is discontinuous at a certain point, we can use limits to find the value that the curve approaches at that point.

Regions

Regions can be bounded or unbounded.

A bounded region means that the line or curve is included in the region.

E.g. Remember that x = 3 is a vertical line with x -intercept 3.

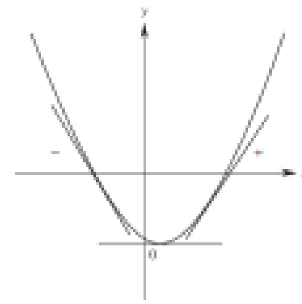
An unbounded region means that the line or curve is not included in the region.

For lines that are not horizontal or vertical, or for curves, we need to check a point to see if it lies in the region.

Sometimes a region includes two or more inequalities. When this happens, sketch each region on the number plane, and the final region is where they overlap (intersect).

The first quadrant is where x and y values are both positive.

Intro to Calculus – Differentiation



Gradient

$$m = \frac{rise}{run}$$

Differentiation from First Principles

DIFFERENTIATION BY FIRST PRINCIPLES

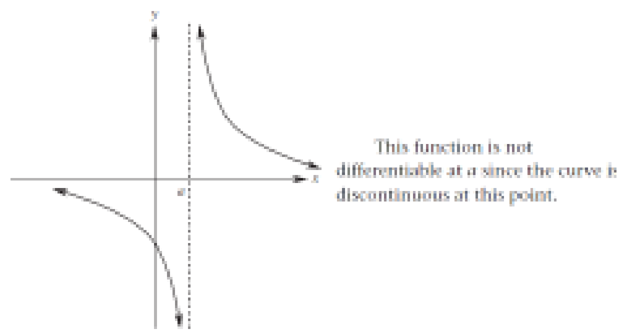
$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The process of finding the gradient of a tangent is called differentiation. The resulting function is called the derivative.

Differentiability

A function is called a differentiable function if the gradient of the tangent can be found.

There are some graphs that are not differentiable in places.



Most functions are continuous, which means that they have a smooth unbroken line or curve. However, some have a gap, or discontinuity, in the graph (e.g. hyperbola).

A function *y = f (x)* is differentiable at the point *x = a* if the derivative exists at that point. This can only happen if the function is continuous and smooth at *x = a*.

DIFFERENTIATION LAWS

$$\frac{dy}{dx} = y' = f'(x)$$

- If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$
- If $y = C$, then $\frac{dy}{dx} = 0$ [where C is a constant]
- If $y = ax^n$, then $\frac{dy}{dx} = anx^{n-1}$
- If $y = f(x) + g(x)$, then $\frac{dy}{dx} = f'(x) + g'(x)$

DIFFERENTIATION METHODS

FUNCTION OF A FUNCTION RULE

If $y = [f(x)]^n$

then $\frac{dy}{dx} = n.f(x)^{n-1}.f'(x)$

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

PRODUCT RULE

If $y = u.v$

then $\frac{dy}{dx} = u'v + uv'$

inside + outside

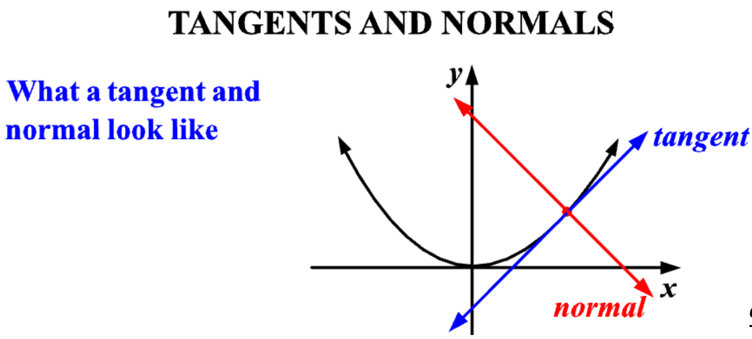
QUOTIENT RULE

If $y = \frac{u}{v}$

then $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$

$$\frac{\text{inside} - \text{outside}}{\text{bottom}^2}$$

Tangents and Normals



$\frac{dy}{dx}$ is the gradient of the tangent to a curve

If lines with gradients m_1 and m_2 are perpendicular, then $m_1 m_2 = -1$